Exercises on Category Theory and Algebra

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Contents

1 Introduction

This version is a very rough draft. Moreover, these notes just contains a more or less related collection of more or less useful exercises/statements/tricks[1] I remembered, heavily biased towards my own interests[2][3]. I will give/reference some Definitions and

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[1] or even answers to questions, that confused me for a time

[2] homotopy theory, algebraic topology, higher algebra

[3] notably, I have avoided exercises on chain complexes, derived functors etc. as covering homological algebra properly – i.e., with stable infinity categories – lies outside of the scope of these notes

Theorems, but rarely motivate or prove them. I will often choose *weirder* definitions, if they generalize to higher categories; So some parts of the exposition won't be standard. You are safe to ignore footnotes and I would advise you to ignore them if you don't know what I'm talking about.

If you have Feedback or Questions, please feel free to talk to me in person or write me an e-mail. In particular, this document almost certainly contains errors – impossible exercises, missing assumptions, typos etc. – I would be very grateful if you let me know.

The difficulty of the exercises will vary[4], some should be very easy, some might be hard. I will write (*) if I think, that the easiest solution using techniques from these notes, is more difficult.

I will use footnotes to indicate fun facts, but you are safe to ignore them. Sometimes knowing previous exercises will make the current exercise easier. To the best of my knowledge, the solution of an exercise I have in mind, does not depend on future exercises. So if you are stuck, I would recommend you to use statements from previous exercises, even if you haven't solved them (yet).

Last but not least, I want to thank Phil Pützstück for his excellent Exercises on Spectra.

Notation and Convention

We will use the following notation

A	a unital (but not necessarily commutative) ring
k	a skew-field
LMod_A	the category of left A-modules
A-module	a left A-module (unless otherwise specified)
1-category	an ordinary category or $(1,1)$-category
category	We write category, if the statement is also true for $(\infty, 1)$ categories
$\mathcal{C}, \mathcal{D}$	categories

I use map and morphism interchangably.

2 Category Theory

2.1 Equivalence of Categories

2.1 Definition. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ be functors. We say that $\mathcal{F}$ and $\mathcal{G}$ are equivalences of categories, if there exist natural isomorphisms $\mathrm{id}_{\mathcal{D}} \Rightarrow \mathcal{F} \circ \mathcal{G}$ and

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[4] especially since I can't predict your background

$\mathrm{id} \Rightarrow \mathcal{G} \circ \mathcal{F}$.

2.2 Exercise. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. Let $\eta\colon \mathcal{F} \to \mathcal{F}'$ be a natural transformation. Show that η is a natural isomorphism if and only if for each object c in $\mathcal{C}$, the map $\eta(c)$ is an isomorphism.
2. Show that a functor $\mathcal{F}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.

2.2 Adjunctions

2.3 Theorem (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category, $c \in \mathcal{C}$ an object and $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ a functor (also called a presheaf). Recall that $\mathrm{Hom}_{\mathcal{C}}(-, c)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is also a functor from $\mathcal{C}^{\mathrm{op}}$ to Set Then there exists a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Set})}(\mathrm{Hom}_{\mathcal{C}}(-, c), \mathcal{F}(-)) \cong \mathcal{F}(c)$$

where the LHS is the set of natural transformations from $\mathrm{Hom}_{\mathcal{C}}(-, c)$ to $\mathcal{F}(-)$.

2.4 Theorem (Yoneda Embedding)**.** *Let $\mathcal{C}$ be a category and $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$, then the Yoneda embedding*

$$\begin{aligned} \text{よ}\colon \mathcal{C} &\to \mathcal{P}(\mathcal{C}) \\ c &\mapsto \mathrm{Hom}_{\mathcal{C}}(-, c) \end{aligned}$$

is a well-defined, fully faithful functor.

2.5 Exercise.

1. Show that the Yoneda-Embedding is well-defined and fully faithful

2.6 Definition (Adjoint Objects)**.** Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}, \mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ be functors.

1. Then an object $c \in \mathcal{C}$ is said to be a right-adjoint object to $d \in \mathcal{D}$ under $\mathcal{F}$ if there exists an natural isomorphism

   $$\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(-), d) \cong \mathrm{Hom}_{\mathcal{C}}(-, c)$$

   of functors from $\mathcal{C}^{\mathrm{op}}$ to Set.
2. Then an object $d \in \mathcal{D}$ is said to be a left-adjoint object to $c \in \mathcal{C}$ under $\mathcal{G}$ if there exists an natural isomorphism

   $$\mathrm{Hom}_{\mathcal{C}}(c, \mathcal{G}(-)) \cong \mathrm{Hom}_{\mathcal{D}}(d, -)$$

   of functors from $\mathcal{C}$ to Set.

2.7 Exercise.

(1) Assume that $c, c' \in \mathcal{C}$ are right-adjoint objects to d under $\mathcal{F}$. Show that c and c' are isomorphic.

(2) Let $\mathcal{C}_1 \xrightarrow{\mathcal{F}_1} \mathcal{C}_2 \xrightarrow{\mathcal{F}_2} \mathcal{D}$ be functors and d in $\mathcal{D}$ an object. Suppose that
   - $c_2 \in \mathcal{C}_2$ is a left (right) adjoint object to d under $\mathcal{F}_2$,
   - $c_1 \in \mathcal{C}_1$ is a left (right) adjoint object to c_2 under $\mathcal{F}_1$.

   Show that c_1 is an adjoint object to $\mathcal{F}_2 \circ \mathcal{F}_1$ under d.

2.8 Definition (Adjoint Functors)**.** Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}, \mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ be functors. We say that $\mathcal{G}$ is a right adjoint functor of $\mathcal{F}$ (and $\mathcal{F}$ a left adjoint functor of $\mathcal{G}$) if there exists a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(-), -) \cong \mathrm{Hom}_{\mathcal{C}}(-, \mathcal{G}(-))$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}$. In that case, $\mathcal{G}$ admits a left adjoint (namely $\mathcal{F}$) and $\mathcal{F}$ admits a right adjoint (namely $\mathcal{G}$) and one often writes $\mathcal{F} \dashv \mathcal{G}$. Moreover, if we have chosen a natural isomorphism $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(-), -) \cong \mathrm{Hom}_{\mathcal{C}}(-, \mathcal{G}(-))$, then the functors $\mathcal{F}, \mathcal{G}$ together with the natural isomorphism is called an adjunction.

2.9 Theorem (Pointwise Construction of Adjoint Functors)**.** *A functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ admits a right (left) adjoint if and only if every $y \in \mathcal{D}$ has a right (left) adjoint object x in $\mathcal{C}$.*

2.10 Exercise (Whiskering)**.** Let $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{C} \to \mathcal{D}$ be functors, $\eta\colon \mathcal{F}_1 \Rightarrow \mathcal{F}_2$ a natural transformation.

(1) Let $\mathcal{G}\colon \mathcal{C}' \to \mathcal{C}$ be a functor. Show that $\eta\mathcal{G}$ is a natural transformation, where we define $(\eta\mathcal{G})(c') = \eta(\mathcal{G}(c'))\colon (\mathcal{F}_1 \circ \mathcal{G})(c') \to (\mathcal{F}_2 \circ \mathcal{G})(c')$.

(2) Let $\mathcal{H}\colon \mathcal{D} \to \mathcal{D}'$ be a functor. Show that $\mathcal{H}\eta$ is a natural transformation, where we define $(\mathcal{H}\eta)(c) = \mathcal{H}(\eta(c))\colon (\mathcal{H} \circ \mathcal{F}_1)(c) \to (\mathcal{H} \circ \mathcal{F}_2)(c)$.

2.11 Theorem (Triangle Identities)**.** *Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}, \mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ be functors. Then an adjunction is equivalently given by*

(1) a natural transformation $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}$, called the unit transformation,

(2) a natural transformation $\epsilon\colon \mathcal{F} \circ \mathcal{G} \Rightarrow \mathrm{id}_{\mathcal{D}}$, called the counit transformation

which satisfy the triangle identities, i.e.,

$$\begin{aligned}(\epsilon\mathcal{F}) \circ (\mathcal{F}\eta) &= \mathrm{id}_{\mathcal{F}} \\ (\mathcal{G}\epsilon) \circ (\eta\mathcal{G}) &= \mathrm{id}_{\mathcal{G}}\end{aligned}$$

agree as natural transformations.

2.12 Exercise.

(1) Prove theorem 2.9.
(2) Prove theorem 2.11.

2.13 Exercise. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor and d in $\mathcal{D}$.

(1) Assume that $\mathcal{G}, \mathcal{G}'\colon \mathcal{D} \to \mathcal{C}$ are right adjoints of $\mathcal{F}$. Show that $\mathcal{G}$ and $\mathcal{G}'$ are naturally isomorphic.
(2) Let

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left, "\mathcal{F}_1"] & \mathcal{C}_2 \arrow[l, bend left, "\mathcal{G}_1"] \arrow[r, bend left, "\mathcal{F}_2"] & \mathcal{C}_3 \arrow[l, bend left, "\mathcal{G}_2"] \end{tikzcd}$$

be adjoint functors. Show that $\mathcal{F}_2 \circ \mathcal{F}_1 \dashv \mathcal{G}_1 \circ \mathcal{G}_2$ is also an adjunction.
(3) Let $\mathcal{F} \dashv \mathcal{G}$ be an adjunction. Show that for any category $\mathcal{E}$
    (a) postcomposition with $\mathcal{F}$ and $\mathcal{G}$ induces an adjunction $\mathcal{F}_* \dashv \mathcal{G}_*$ on functor categories $\mathrm{Fun}(\mathcal{E}, \mathcal{C}), \mathrm{Fun}(\mathcal{E}, \mathcal{D})$.
    (b) precomposition with $\mathcal{F}$ and $\mathcal{G}$ induces an adjunction $\mathcal{G}_* \dashv \mathcal{F}_*$ on functor categories $\mathrm{Fun}(\mathcal{C}, \mathcal{E}), \mathrm{Fun}(\mathcal{D}, \mathcal{E})$.

2.3 Limits and Colimits

2.14 Exercise.

(1) Let $\mathcal{F}\colon \mathcal{C}' \to \mathcal{C}, \mathcal{G}\colon \mathcal{D} \to \mathcal{D}'$ be functors and $\mathcal{E}$ another category. Show that
    (a) precomposition with $\mathcal{F}$ induces a functor $\mathcal{F}^*\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}', \mathcal{D})$,
    (b) postcomposition with $\mathcal{F}$ induces a functor $\mathcal{G}_*\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}')$.
    (c) pre- and postcomposition commute, that is $\mathcal{F}^* \circ \mathcal{G}_* = \mathcal{G}_* \circ \mathcal{F}^*$.
(2) Let $*$ be the terminal category, i.e. the category with a single object and the identity as sole morphism. Observe that there exists a unique functor $1\colon \mathcal{C} \to *$ for each category $\mathcal{C}$.
(3) Show that there exists an equivalence of categories $\mathrm{Fun}(*, \mathcal{C}) \simeq \mathcal{C}$. Conclude that there exists a functor $\mathrm{const}\colon \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ obtained by precomposition with $\mathcal{C} \to *$.

2.15 Definition. Let $\mathcal{I}$ be a category and $I\colon \mathcal{I} \to \mathcal{C}$ a functor. Recall that precomposition with $I \to *$ induces a functor $\mathrm{const}\colon \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$. Then a limit/colimit is a right-adjoint/left-adjoint object $\lim_I /\ \mathrm{colim}_I$ to const under I^* together with a fixed natural isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(\mathrm{const}(-), I) &\cong \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(-, \lim_I) \\ \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(I, \mathrm{const}(-)) &\cong \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(\operatorname*{colim}_I, -) \end{aligned}$$

2.16 Exercise.

1. Let $I\colon \mathcal{I} \to \mathcal{C}$. Show that a limit/colimit $\lim_I/\operatorname{colim}_I$ is – if it exists – unique up to unique isomorphism .
2. Let $I\colon \mathcal{I} \to \mathcal{C}, \mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ be functors. Assume that $\lim_I, \lim_{\mathcal{F}\circ I}$ and $\operatorname{colim}_I, \operatorname{colim}_{\mathcal{F}\circ I}$ exist. Show that there exists a canonical map $\mathcal{F}(\lim_I) \to \lim_{\mathcal{F}\circ I}, \operatorname{colim}_{\mathcal{F}\circ I} \to \mathcal{F}(\operatorname{colim}_I)$.
3. Show that limits in $\mathcal{C}^{\mathrm{op}}$ correspond to colimits in $\mathcal{C}$ and vice versa .

2.17 Exercise (Some (Co-)Limits in Set and Cat)**.**

1. Let I be a discrete category[5] and $a\colon I \to \mathrm{Set}$ a functor. Show that $\lim_I \cong \prod_{i\in I} a_i$ is the cartesian product.
2. Let I be a discrete category and $a\colon I \to \mathrm{Set}$ a functor. Show that $\operatorname{colim}_I \cong \coprod_{i\in I} a_i$ is the disjoint union.
3. Let I be a discrete category and $a\colon I \to \mathrm{Cat}_{(1,1)}$ a functor. Give an explicit description of $\lim_I \cong \prod_{i\in I} a_i$.
4. Let I be a discrete category and $a\colon I \to \mathrm{Cat}_{(1,1)}$ a functor. Give an explicit description of $\operatorname{colim}_I \cong \coprod_{i\in I} a_i$.
5. Let $X \overset{f_2}{\underset{f_1}{\rightrightarrows}} Y$ be maps between sets, which we regard as functor from the category $0 \rightrightarrows 1$.
    1. (a) give an explicit description of the limit
    2. (b) give an explicit description of the colimit

2.18 Definition. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $I\colon \mathcal{I} \to \mathcal{C}$ be functors. Assume that $\lim_I$ (colim_I) exists. Then $\mathcal{F}$ preserves the limit (colimit) of I, if the canonical map from item (2) is an isomorphism.

2.19 Definition. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $I\colon \mathcal{I} \to \mathcal{C}$ be functors. Then $\mathcal{F}$ is said to reflect the limit of I, if for each object c in $\mathcal{C}$ and each natural transformation $I \Rightarrow \mathrm{const}(c)$, such that $\mathcal{F}\circ I \Rightarrow \mathcal{F}\circ \mathrm{const}(c) \simeq \mathrm{const}(\mathcal{F}(c))$ exhibits $\mathcal{F}(c)$ as limit of $\mathcal{F}\circ I$, the natural transformation $I \Rightarrow \mathrm{const}(c)$ already exhibited c as limit of I. Moreover, we say that $\mathcal{F}$ reflects limits, if it reflects the limit of any small diagram $I : \mathcal{I} \to \mathcal{C}$. Furthermore, there exists a dual notaion of functors reflecting colimits (of I).

2.20 Exercise.

1. Show that any fully faithful functor reflects limits and colimits.
2. Show that $\mathrm{Hom}_{\mathcal{C}}(c,-)\colon \mathcal{C} \to \mathrm{Set}, \mathrm{Hom}_{\mathcal{C}}(-,c)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ preserve limits[6].

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[5] that is a category where each morphism is an identity morphism – one can also call I a set

[6] Recall that limits in $\mathcal{C}^{\mathrm{op}}$ are colimits in $\mathcal{C}$ (cf. exercise 2.16, (3))

(3) Show that the Yoneda embedding $\text{よ} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ preserves and reflects limits. Conclude that for any category $\mathcal{C}$, there exists a limit preserving and limit reflecting, fully faithful functor into a bicomplete category[7].

(4) Show that for any category $\mathcal{C}$, there exists a colimit preserving and colimit reflecting, fully faithful functor into a bicomplete category.

2.21 Theorem. *Left adjoint functors preserve colimits and right adjoint functors preserve limits.*

2.22 Exercise.

(1) Show theorem 2.21

2.4 Kan Extensions

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories and $f : \mathcal{C} \to \mathcal{D}$ a functor. One can ask, whether there is a canonical way to extend any functor $\mathcal{G} : \mathcal{C} \to \mathcal{E}$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathcal{G}"] \arrow[d, "f"'] & \mathcal{E} \\
\mathcal{C}' \arrow[ur, dashed, "?"] &
\end{tikzcd}$$

unfortunately, there is in general no way to extend $\mathcal{G}$ while making the diagram above commute. Nonetheless, there are two possibilities – left and right Kan extension – to define a canonical extension:

2.23 Definition (Kan Extension)**.** Recall that precomposition with f induces a functor $f^* : \mathrm{Fun}(\mathcal{C}', \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$. Let $\mathcal{G} : \mathcal{C} \to \mathcal{E}$ be an object in $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$, then the left Kan extension $\mathrm{Lan}_f(\mathcal{G}) : \mathcal{C}' \to \mathcal{E}$ of $\mathcal{G}$ along f – if it exists – is defined as left adjoint object of $\mathcal{G}$ under f^*. Similarly, the right Kan extension $\mathrm{Ran}_f(\mathcal{G}) : \mathcal{C}' \to \mathcal{E}$ of $\mathcal{G}$ under f^* – if it exists – is defined as right adjoint object of $\mathcal{G}$ under f^*.

2.24 Theorem (Kan Extension is Functorial)**.** *Let $f : \mathcal{C} \to \mathcal{C}'$ be a functor and assume that all left (right) Kan extensions of functors $\mathcal{G} : \mathcal{C} \to \mathcal{D}$ exist. Then there exists a functor*

$$\begin{aligned}
\mathrm{Lan}_f(-) : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) &\to \mathrm{Fun}(\mathcal{C}', \mathcal{D}) \\
\mathcal{G} &\mapsto \mathrm{Lan}_f(\mathcal{G})
\end{aligned}$$

2.25 Theorem (Associativity of Kan Extensions)**.** *Let $\mathcal{C}_1 \xrightarrow{f_1} \mathcal{C}_2 \xrightarrow{f_2} \mathcal{C}_3$ and $\mathcal{G} : \mathcal{C}_1 \to \mathcal{E}$ be functors. Suppose that $\mathrm{Lan}_{f_1}(\mathcal{G})$ and $\mathrm{Lan}_{f_2}(\mathrm{Lan}_{f_1}(\mathcal{G}))$ exists. Then so does $\mathrm{Lan}_{f_2 \circ f_1}(\mathcal{G})$ and it naturally equivalent to $\mathrm{Lan}_{f_2}(\mathrm{Lan}_{f_1}(\mathcal{G}))$*

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[7] This is a standard trick to reduce a question about limits to a category with all small limits

2.26 Exercise.

(1) Show that left (right) Kan extensions are unique up to natural equivalence.
(2) Show theorem 2.24
(3) Show theorem 2.25

2.27 Exercise.

(1) Let $r : \mathcal{I} - > *$ be the functor to the terminal category (see e.g.exercise 2.14(2)) and $\mathcal{F} : \mathcal{I} \to \mathcal{C}$. Show that left (right) Kan extension of $\mathcal{F}$ along r is canonically isomorphic to $\operatorname{colim} \mathcal{F}$ $(\lim \mathcal{F})$.
(2) Let $f : \mathcal{I} \to \mathcal{J}$ be a functor and $\mathcal{F} : \mathcal{I} \to \mathcal{C}$ any diagram. Show that $\lim_{\mathcal{J}} \operatorname{Ran}_f(\mathcal{F}) \simeq \lim_{\mathcal{I}} \mathcal{F}$ (and $\operatorname{colim}_{\mathcal{J}} \operatorname{Lan}_f(\mathcal{F}) \simeq \operatorname{colim}_{\mathcal{I}} \mathcal{F}$) are canonically isomorphic.

2.28 Definition (Slice Category)**.** Let $\operatorname{Ar}(\mathcal{C}) = \operatorname{Fun}(\Delta^1, \mathcal{C})$ be the arrow category and $c \in \mathcal{C}$ be an object. Recall that the Arrow category admits a target (source) functor $t : \operatorname{Ar}(\mathcal{C}) \to \mathcal{C}$ given by evaluating at the target (source). Then the slice (coslice) category $\mathcal{C}_{/c}$ $(\mathcal{C}_{c/})$ is defined as pullback in Cat on the left (right)

$$\begin{tikzcd}
\mathcal{C}_{/c} \arrow[r] \arrow[d] & \operatorname{Ar}(\mathcal{C}) \arrow[d, "t"] \\
* \arrow[r, "c"] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_{c/} \arrow[r] \arrow[d] & \operatorname{Ar}(\mathcal{C}) \arrow[d, "s"] \\
* \arrow[r, "c"] & \mathcal{C}
\end{tikzcd}$$

2.29 Exercise.

(1) Show that $\mathcal{C}_{/c}$ has objects of the form $f : x \to c$ and morphisms $g : (f_1 : x_1 \to c) \to (f_2 : x_2 \to c)$ are given by morphisms $g : x_1 \to x_2$ making the evident triangle commute.
(2) Give an explicit description of $\mathcal{C}_{c/}$.
(3) Give an informal description of the forgetful functor $\mathcal{C}_{c/} \to \operatorname{Ar}(\mathcal{C}) \xrightarrow{t} \mathcal{C}$ and $\mathcal{C}_{/c} \to \operatorname{Ar}(\mathcal{C}) \xrightarrow{s} \mathcal{C}$.

2.30 Definition. Now let $\mathcal{F} : \mathcal{D} \to \mathcal{C}$ be a functor. We define $\mathcal{F}_{/c}$ $(\mathcal{F}_{c/})$ as pullback.

$$\begin{tikzcd}
\mathcal{F}_{/c} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "\mathcal{F}"] \\
\mathcal{C}_{/c} \arrow[r] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_{c/} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "\mathcal{F}"] \\
\mathcal{C}_{c/} \arrow[r] & \mathcal{C}
\end{tikzcd}$$

2.31 Exercise.

(1) Give an informal description of $\mathcal{F}_{/c}$ and $\mathcal{F}_{c/}$.

2.32 Theorem (Kan Extension Formula[8]). *Assume we have functors*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathcal{F}"] \arrow[d, "f"] & \mathcal{D} \\
\mathcal{C}' &
\end{tikzcd}$$

and that all following colimits exist. Then $\mathrm{Lan}_f\mathcal{F}$ *exists and is given by*

$$\mathrm{Lan}_f\mathcal{F}(x) \simeq \operatorname*{colim}_{y, f(y)\to x} \mathcal{F}(y)$$

Here $y, x \to f(x)$ *is more formally given by the category* $\mathcal{F}_{/x}$*, i.e. the category containing morphisms* $f(y) \xrightarrow{h} x$ *as objects and a morphism* $(f(y) \to x) \to (f(y') \to x)$ *is given by a morphism* $g : y \to y'$ *making the diagram*

$$\begin{tikzcd}
f(y) \arrow[rr, "f(g)"] \arrow[dr, "h"] & & f(y') \arrow[dl, "h'"'] \\
& x &
\end{tikzcd}$$

Similarly, if all of the following limits exist, then so does $\mathrm{Ran}_f\mathcal{F}(x)$ *and is given by*

$$\mathrm{Ran}_f\mathcal{F}(x) \simeq \lim_{y, x\to f(y)} \mathcal{F}(y)$$

2.33 Exercise.

(1) Let $\mathcal{I}, \mathcal{J}$ be categories and $\mathcal{F} : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ a functor. Show that $\lim_{\mathcal{I}\times\mathcal{J}} \mathcal{F}(i,j)$ and $\lim_{\mathcal{I}} \lim_{\mathcal{J}}(\mathcal{F}(i,j))$ are canonically isomorphic.

(2) Conclude that limits (colimits) commute with limits (colimits).

2.34 Definition (Cofinal and Initial Functors[9]). Let $p : \mathcal{I} \to \mathcal{J}$ be a functor between small[10] categories. We call p 1-initial (1-coinitial)[11] if for each functor $\mathcal{F} : \mathcal{J} \to \mathrm{Set}$, the canonical map $\lim_{\mathcal{J}} \mathcal{F} \to \lim_{\mathcal{I}} \mathcal{F}\circ p$ ($\mathrm{colim}_{\mathcal{I}} : \mathcal{F}\circ p \to \mathrm{colim}_{\mathcal{J}} \mathcal{F}$) is an isomorphism.

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[8] to the best of my knowledge, the only proof I understand uses straightening-unstraightening

[9] there are many possible conventions (any combination of (co-)final and (co-)initial is used by someone), but I like this one: the inclusion of an *initial* object is *initial* and *co*limits behave well with *co*initial functors

[10] you may even in practice ignore size issues

[11] this definition also works by replacing Set with “∞-Set“ $= \infty - \mathrm{Gpd} = \mathrm{An}$

2.35 Exercise. Let $p : \mathcal{I} \to \mathcal{J}$ be an 1-initial (1-coinitial) functor between small categories and $\mathcal{F} : \mathcal{I} \to \mathcal{F}$ another functor. Show that

- (1) $\lim_{\mathcal{J}} \mathcal{F}$ ($\operatorname{colim}_{\mathcal{J}} \mathcal{F}$) exists if and only if $\lim_{\mathcal{I}} \mathcal{F} \circ p$ ($\operatorname{colim}_{\mathcal{I}} \mathcal{F} \circ p$)
- (2) In that case, both limits (colimits) are canonically isomorphic.

2.36 Theorem (Quillen Theorem A)**.** *Let $p : \mathcal{I} \to \mathcal{J}$ be a functor. Then p is 1-initial (1-coinitial) if and only if for each object $j \in J$, the category $p_{/j}$ ($p_{j/}$) is nonempty and connected*[12] *(cf. definition 2.30).*

2.37 Exercise.

- (1) Show that any left (right) adjoint $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a 1-initial (1-coinitial) functor.
- (2) Let $\mathcal{C}_3 \xrightarrow{\mathcal{F}_1} \mathcal{C}_2 \xrightarrow{\mathcal{F}_2} \mathcal{C}_1$ be functors.
  - (a) Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are 1-initial (1-coinitial). Show that their composition is also 1-initial (1-coinitial).
  - (b) Suppose that $\mathcal{F}_2 \circ \mathcal{F}_1$ and $\mathcal{F}_2$ are initial (coinitial). Show that $\mathcal{F}_1$ is also 1-initial (1-coinitial).

2.38 Exercise (Fun with Kan Extensions)**.** These are lemmas we find very useful. We advise you to prove them with Kan extensions[13] but you can also prove them elementary.

- (1) Let

$$\begin{tikzcd}
c_1 \arrow[r] \arrow[d] & c_2 \arrow[r] \arrow[d] & c_3 \arrow[d] \\
d_1 \arrow[r] & d_2 \arrow[r] & d_3
\end{tikzcd}$$

  be a commutative diagram. Show
  - (a) If both inner squares are pullbacks, then so is the outer square.
  - (b) If both inner squares are pushouts, then so is the outer square.
  - (c) If the outer square and the right inner square are pullbacks, then so is the left inner square.
  - (d) If the outer square and the left inner square are pushouts, then so is the right inner square.
- (2) Let

$$\begin{tikzcd}
c \arrow[r] \arrow[d, dashed] & d \arrow[d, "\cong"] \\
c' \arrow[r] & d'
\end{tikzcd}$$

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[12] for the higher categorical definition, replace nonempty and connected with weakly contractible

[13] Kan extensions work roughly the same for higher categories and are necessary to prove this statement coherently

be a commutative square, where the right leg is an isomorphism. Show that it is a pullback square if and if the left leg is an isomorphism.

(3) Formulate the dual statement to the previous exercise.

2.39 Definition (Pointed Category and Zero Object)**.** A category $\mathcal{C}$ is said to be pointed, if it admits an object 0 which is both initial and terminal. In this case, 0 is called the zero object.

2.40 Definition (Fiber and Cofiber)**.** Assume that $\mathcal{C}$ admits finite limits (colimits) and a zero object[14]. Let $f : c \to c'$ be a morphism. We define the fiber $\operatorname{fib}(f)$ of f as the pullback

$$\begin{tikzcd}
\operatorname{fib}(f) \arrow[r] \arrow[d] & c \arrow[d, "f"] \\
* \arrow[r] & c'
\end{tikzcd}$$

where $*$ is the zero object. Dually, the cofiber $\operatorname{cofib}(f)$ of f is the pushout

$$\begin{tikzcd}
c \arrow[r, "f"] \arrow[d] & c' \arrow[d] \\
* \arrow[r] & \operatorname{cofib}(f)
\end{tikzcd}$$

2.41 Exercise (Some Corollaries)**.**

(1) Let $c \xrightarrow{f} c' \xrightarrow{g} d$ be two morphisms. Show that there exists a canonical isomorphism $d/(c') \cong (d/c)/(c'/c)$ where we write c'/c for $\operatorname{cofib}(f)$ and so on.

(2) Let

$$\begin{tikzcd}
c_1 \arrow[r] \arrow[d] & c_2 \arrow[r] \arrow[d] & c_3 \arrow[d] \\
d_1 \arrow[r] & d_2 \arrow[r] \arrow[d] & d_3 \arrow[d] \\
 & e_1 \arrow[r] & e_2
\end{tikzcd}$$

be a commutative diagram, where the top left square, top outer square and the bottom right square are pushouts. Show that every square in this diagram is a pushout.

---

[14] the definition of a fiber is also useful for a category, which just admits a terminal object. However the dual version, that is a pushout with initial object, is not useful, since there exists only one map into the empty set

(3) Let $c_1, c_2 \to d$ and $d \to e$ be morphisms. Show that there exists a pullback square

$$\begin{tikzcd}
c_1 \times_d c_2 \arrow[r] \arrow[d] & c_1 \times_e c_2 \arrow[d] \\
d \arrow[r] & d \times_e d
\end{tikzcd}$$

(4) Let

$$\begin{tikzcd}
c_1 \arrow[r] \arrow[d] & c_2 \arrow[r] \arrow[d] & c_3 \arrow[d] \\
d_1 \arrow[r] \arrow[d] & d_2 \arrow[r] \arrow[d] & d_3 \arrow[d] \\
e_1 \arrow[r] & e_2 \arrow[r] & e_3
\end{tikzcd}$$

be a diagram. Show that that the colimit of this diagram coincides with the colimit one obtains by first taking the colimit of the rows and then the colimit of the resulting pushout.

(5) Let $\mathcal{C}$ be a pointed category and

$$\begin{tikzcd}
c_1 \arrow[r] \arrow[d] & c_2 \arrow[r] \arrow[d] & c_3 \arrow[d] \\
d_1 \arrow[r] \arrow[d] & d_2 \arrow[r] \arrow[d] & d_3 \arrow[d] \\
e_1 \arrow[r] & e_2 \arrow[r] & e_3
\end{tikzcd}$$

be a diagram, where all three rows and $c_1 \to d_1 \to e_1, c_2 \to d_2 \to e_2$ are cofiber sequences. Show that $c_3 \to d_3 \to e_3$ is also a cofiber sequence.

3 Some Applications

3.1 Filtered Colimits, Compact Objects

3.1 Definition (strictly finite category)**.** We call a category $\mathcal{C}$ strictly finite if it has a finite set of objects and if the Hom-Set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is a finite set for each pair of objects x, y. By abuse[15] of notation, we will call a strictly finite category just finite.

---

[15] We don't think that this is the "correct" definition. It "works" for $(1,1)$-categories, but does not work (naively) for infinity categories. For example, $\mathrm{B}\mathbb{Z}$ (cf. Exercise 3.16 (2)) is actually finite (for the "correct" definition) category, whereas $\mathrm{B}\mathbb{Z}/2\mathbb{Z}$ is not.

3.2 Definition (right cone)**.** Let $\mathcal{C}$ be a category. Then the right cone is the category $\mathcal{C}^{\triangleright}$ containing

(1) $\mathrm{Ob}(\mathcal{C}^{\triangleright}) = \mathrm{Ob}(\mathcal{C}) \coprod \{*\}$,
(2) and

$$\mathrm{Hom}_{\mathcal{C}}(x, y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(x, y) & \text{if } x, y \in \mathrm{Ob}(\mathcal{C}) \\ \{1_x\} & \text{if } y = * \\ \emptyset & \text{else} \end{cases}$$

In other words, we add an object $*$ for each object $x \in \mathcal{C}$, we add a unique morphism $1_x \colon x \to *$. Observe that there exists a canonical fully faithful functor $\mathcal{C} \hookrightarrow \mathcal{C}^{\triangleright}$

3.3 Definition (filtered category)**.** We call a category I filtered, if each finite category $\mathcal{C}$ and each functor $I \to \mathcal{C}$, there exists an extension

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & I \\
\mathcal{C}^{\triangleright} \arrow[ur, dashed] &
\end{tikzcd}$$

3.4 Exercise.

(1) Show that a category I is filtered if and only if the following holds:
  (a) for two objects $x, y \in I$, there exists an object z and morphisms $x \to z, y \to z$,
  (b) for two morphisms $f, g \colon x \to y$ there exists an object z and a morphism $h \colon y \to z$ such that $h \circ g = h \circ f$.
(2) Let I be a category with finite colimits. Show that I is a filtered category.

3.5 Theorem (filtered colimits are exact)**.** *Let I be a small category. Then the following are equivalent:*

(1) I is a filtered category.
(2) the functor $\mathrm{colim} \colon \mathrm{Fun}(I, \mathrm{Set}) \to \mathrm{Set}$ *preserves finite limits.*

3.6 Definition (Compact Object)**.** We say that an object c in $\mathcal{C}$ is compact[16] if the functor $\mathrm{Hom}_{\mathcal{C}}(c, -)$ preserves filtered colimits.

3.7 Exercise.

---

[16] Surprisingly, compact topological spaces are not compact objects in the category Top, see e.g. this MathOverflow Post. In short, an object in Top is a compact object if and only if it is a finite discrete space.

(1) Show that compact objects are closed under finite colimits.
(2) Show that compact objects are closed under retracts.
(3) Let c be a compact object, which is the filtered colimit $\operatorname{colim}_I c_i$. Show that c is a retract of some c_i.

3.8 Exercise (Compact Objects)**.**

(1) Show that the compact objects in Set are precisely the finite sets.
(2) Show that the compact objects in Mon are precisely the finitely generated Monoids.
(3) Show that the compact objects in CMon are precisely the finitely generated commutative Monoids.
(4) Show that the compact objects in Grp are precisely the finitely generated Groups.
(5) Show that the compact objects in Ab are precisely the finitely generated abelian groups.
(6) Show that the compact objects in LMod_A are precisely the finitely generated A-modules
(7) Show that the compact objects in Ring are precisely finite type rings.
(8) Show that the compact objects in CRing are precisely the finite type commutative rings.
(9) Show that the compact objects in Alg_A are precisely the finite type A-Algebras.
(10) Show that the compact objects in CAlg_A are precisely the finite type commutative A-Algebras.

3.2 Monomorphisms and Epimorphisms

Let $f\colon c \to c'$ be a morphism. We say that f is a monomorphism/epimorphism if the left/right square

$$\begin{tikzcd}
c \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & c \arrow[d] \\
c \arrow[r] & c'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
c \arrow[r] \arrow[d] & c' \arrow[d, "\mathrm{id}"] \\
c' \arrow[r, "\mathrm{id}"] & c'
\end{tikzcd}$$

is a pullback/pushout.

3.9 Exercise.

(1) Show that the composition of monomorphism/epimorphisms is a monomorphism/epimorphism.
(2) Let $c \xrightarrow{f} d \xrightarrow{g} e$ be morphisms, such that $g \circ f$ is a monomorphism/epimorphism. Show that f/g is also a monomorphism/epimorphism.

(3) Show that a morphism $f\colon c \to c'$ is an epimorphism if and only if for every pair of morphism $g, g'\colon c' \to d$ it follows: $g \circ f = g' \circ f \Rightarrow g = g'$.

(4) Show that a morphism $f\colon c \to c'$ is a monomorphism if and only if for every pair of morphism $g, g'\colon d \to c$ it follows: $f \circ g = f \circ g' \Rightarrow g = g'$.

(5) Conclude that monomorphisms/epimorphisms in Set are precisely injections/surjections.

(6) Show that any isomorphism is a monomorphism/epimorphism.

(7) Show that the inclusion $\mathbb{Z} \to \mathbb{Q}$ is both a monomorphism and epimorphism in category of commutative rings CRing. Conclude that a morphism, which is both a monomorphism and epimorphism is in general not an isomorphism.

3.3 Reflective Subcategories

Let $\mathcal{F} \dashv \mathcal{G}$ be an adjunction, where $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is the left adjoint.

3.10 Definition. We call

(1) $\mathcal{C}$ a reflective subcategory if $\mathcal{G}$ is fully faithful

(2) $\mathcal{D}$ a coreflective subcategory if $\mathcal{F}$ is fully faithful

3.11 Exercise.

(1) Show that $\mathcal{C}$ is a reflective subcategory if and only if the counit is a natural isomorphism.

(2) Show that $\mathcal{D}$ is a coreflective subcategory if and only if the unit is a natural isomorphism.

(3) Assume $\mathcal{F}$ and $\mathcal{G}$ are fully faithful. Then show that both are equivalences of categories.

3.12 Exercise. Let $\mathcal{C}$ be a reflective subcategory of $\mathcal{D}$.

(1) Observe that Ab is a reflective subcategory of Grp

(2) Determine a formula for colimits in $\mathcal{C}$ depending on $\mathcal{F}, \mathcal{G}$ and colimits in $\mathcal{D}$.

3.13 Definition (S-Local Functor)**.** Let $\mathcal{C}, \mathcal{D}$ be categories and let $S \subseteq \mathrm{Mor}(\mathcal{C})$ be a collection of morphisms. Then a functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is said to be S-local, if $\mathcal{F}(f)$ is an isomorphism for $f \in S$. Moreover, we call an object c in $\mathcal{C}$ S-local, if the presheaf $\mathrm{Hom}_{\mathcal{C}}(-, c)$ is S-local.

3.14 Exercise. Let $\mathcal{C}$ be a reflective subcategory of $\mathcal{D}$ and let $S := \{\eta(d)\}$ be the collection of unit morphisms.

Show that an object $d \in \mathcal{D}$ is S-local if and only if it lies in the essential image of the inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$.

Show that the collection of S-local objects is closed under limits. Conclude that $\mathcal{C} \subseteq \mathcal{D}$ is closed under limits, which exist in $\mathcal{D}$.

Conclude that $\mathcal{C}$ is bicomplete if $\mathcal{D}$ is.

3.4 Groupoids

3.15 Definition. We call a category X a groupoid[17], if every morphism is an isomorphism. We write $\mathrm{Gpd}_{(1,1)}$ for the full subcategory of category $\mathrm{Cat}_{(1,1)}$ containing groupoids.

3.16 Exercise.

(1) Show that there exists a fully faithful inclusion $\mathrm{Set} \hookrightarrow \mathrm{Gpd}_{(1,1)}$

(2) Let G be a group, then we write $\mathrm{B}G$ for the category containing a single object $*$ and $\mathrm{Hom}_{\mathrm{B}G}(*, *) = G$ with multiplication as composition.

   (a) Convince yourself, that $\mathrm{B}G$ is a well-defined category.

   (b) Show that there exists a functor $\mathrm{B}(-)\colon \mathrm{Grp} \to \mathrm{Gpd}_{(1,1)}$ which maps a group G to $\mathrm{B}G$.

   (c) Show that this functor $\mathrm{B}(-)$ is fully faithful[18]

   (d) Let X, Y be groupoids and $\mathcal{F}\colon X \to Y$ an adjoint functor. Show that $\mathcal{F}$ is an equivalence.

4 Category of Modules

4.1 Exercise.

(1) Let M be an A-Module. Show that the functors

   (a) $\mathrm{Hom}_{\mathrm{LMod}_A}(M, -)\colon \mathrm{LMod}_A \to \mathrm{Set}$

   (b) $\mathrm{Hom}_{\mathrm{LMod}_A}(-, M)\colon \mathrm{LMod}_A^{\mathrm{op}} \to \mathrm{Set}$

   factor canonically through Ab.

(2) Show that the forgetful functors $\mathrm{Forget}\colon \mathrm{LMod}_A \to \mathrm{Ab}, \mathrm{Forget}\colon \mathrm{LMod}_A \to \mathrm{Set}$ are naturally equivalent to $\mathrm{Hom}_{\mathrm{LMod}_A}(A, -)\colon \mathrm{LMod}_A \to \mathrm{Ab}, \mathrm{Hom}_{\mathrm{LMod}_A}(A, -)\colon \mathrm{LMod}_A \to \mathrm{Set}$.

(3) Show that both forgetful functors above

   (a) preserve limits,

   (b) reflect limits,

   (c) are conservative.

(4) Show that the forgetful functor $\mathrm{Forget}\colon \mathrm{LMod}_A \to \mathrm{Ab}$

   (a) preserves colimits

---

[17] Groupoids are very important in algebraic topology: roughly speaking, there exists an equivalence of categories between "Spaces up to homotopy" and infinity groupoids = "∞-Sets "(= Anima)

[18] If we would work higher categorical, then we would have an equivalence $\mathrm{Grp}(\infty - \mathrm{Gpd}) \cong \infty - \mathrm{Gpd}_{\geq 1,*}$ of groupoid/anima valued groups and pointed, connected groupoids/anima. This equivalence is important in homotopy theory and for example can be leveraged to prove the classical Seifert-van-Kampen for free.

    (b) reflects colimits

(5) Show that LMod_A is bicomplete.

(6) Show that LMod_A admits a zero object.

(7) Let M_1, M_2 be A-modules. Show that the canonical map

$$M_1 \coprod M_2 \xrightarrow{\begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}} M_1 \times M_2$$

is an isomorphism. We write $M_1 \oplus M_2$ for $M_1 \coprod M_2 \cong M_1 \times M_2$.

(8) Let M_1, M_2 be A-modules. Show that the map

$$M_1 \coprod M_2 \xrightarrow{\begin{pmatrix} \mathrm{id} & \mathrm{id} \\ 0 & \mathrm{id} \end{pmatrix}} M_1 \times M_2$$

is an isomorphism.

(9) Show that a morphism $f\colon M \to M'$ in LMod_A is a monomorphism/epimorphism if and only if it is an injection/surjection.

(10) Show that a morphism $f\colon G \to G'$ in Grp is a monomorphism[19] if and only if it is an injection.

4.2 Exercise.

(1) Show that the canonical inclusions

    (a) $\mathrm{LMod}_{\mathbb{Q}} \to \mathrm{Ab}$

    (b) $\mathrm{LMod}_{\mathbb{F}_p} \to \mathrm{Ab}$

are fully faithful[20]. You deserve bonus points, if you use universal properties.

(2) Let $C(\mathrm{LMod}_A) = \mathrm{Hom}_{\mathrm{Fun}(\mathrm{LMod}_A, \mathrm{LMod}_A)}(\mathrm{id}, \mathrm{id})$ be the set of natural transformations from the identity functor to itself. Show that $C(\mathrm{LMod}_A)$ is canonically isomorphic to the center of A.

4.1 Compactness

4.3 Exercise.

(1) Show that an A-module is finitely generated if and only if it is a compact object in LMod_A

(2) Let A be a ring. Show that the following are equivalent:

---

[19] epimorphisms in Grp are also precisely surjections, see e.g. this nLab article

[20] the case for $\mathbb{Q}$ is also true for the higher categorical version of modules, the second case however fails for the inclusion of derived categories $\mathcal{D}(\mathbb{F}_p) \to \mathcal{D}(\mathbb{Z})$ or inclusion of $\mathbb{F}_p$ module spectra into spectra

  - (a) the set of noetherian modules coincides with the set of compact objects,
  - (b) A is a noetherian ring,
  - (c) A is a noetherian module,
  - (d) Subobjects of compact objects are compact.
- (3) Show that there exists a ring A and a noetherian A-module M which is not compact.
- (4) Show that there exists a ring A and a compact A-module M which is not noetherian.
- (5) Show that there exists a ring A, a compact A-module M and a submodule N which is not compact.

5 Exactness

Throughout this section, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ will denote abelian categories. But you can also assume that $\mathcal{A} \in \{\mathrm{LMod}_A, \mathrm{LMod}_A^{\mathrm{op}}\}, \mathcal{B} \in \{\mathrm{LMod}_B, \mathrm{LMod}_B^{\mathrm{op}}\}, \mathcal{C} \in \{\mathrm{LMod}_C, \mathrm{LMod}_C^{\mathrm{op}}\}$ for rings A, B, C.

5.1 Exercise. Let $f\colon M \to N$ be a module homomorphism of A-modules. Show that $\mathrm{im}(f) \cong M/\ker(f)$ and $\mathrm{im}(f) \cong \ker(N \to \mathrm{coker}(f))$ are canonically isomorphic.

5.2 Definition.

- (1) We call a sequence $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ in $\mathcal{A}$ short exact if it fits into a square

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d] & M_2 \arrow[d] \\ 0 \arrow[r] & M_3 \end{tikzcd}$$

  which is both a pushout and a pullback.
- (2) Let $N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3$ be a sequence in $\mathcal{A}$. Then we call this sequence exact at N_2, if the canonical sequence $0 \to \ker(g) \to N_2 \to \mathrm{coker}(f) \to 0$ is short exact.
- (3) Let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be a functor. We say that $\mathcal{F}$ is
  - (a) additive, if it preserves finite products and/or[21] coproducts
  - (b) left-exact, if it preserves finite limits,
  - (c) right-exact, if it preserves finite colimits,
  - (d) exact, if it is both left- and right-exact (or preserves finite limits and colimits).

5.3 Exercise.

---

[21] cf. exercise 4.1 (7)

(1) Let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be an additive functor. Show that for any object M, the morphism $n\cdot\colon M \to M$ maps to the morphism $n\cdot\colon \mathcal{F}(M) \to \mathcal{F}(M)$

(2) Let $\mathcal{F}(-,-)\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a functor, which is additive in both variables. Show that $\mathcal{F}(\cdot p, \mathrm{id}) = \mathcal{F}(\mathrm{id}, \cdot p)$ coincide.

(3) Show that a functor $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ is exact if and only if it preserves short exact sequences.

(4) Let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be a functor. Then show that the following are equivalent:
   (a) $\mathcal{F}$ is a right-exact functor.
   (b) If $M_1 \to M_2 \to M_3 \to 0$ is a sequence, which is exact at M_2 and M_3, then $\mathcal{F}(M_1) \to \mathcal{F}(M_2) \to \mathcal{F}(M_3) \to \mathcal{F}(0)$ is also exact at $\mathcal{F}(M_2)$ and $\mathcal{F}(M_3)$.

(5) Let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be a functor. Then show that the following are equivalent:
   (a) $\mathcal{F}$ is a left-exact functor.
   (b) If $0 \to M_1 \to M_2 \to M_3$ is a sequence, which is exact at M_1 and M_2, then $\mathcal{F}(0) \to \mathcal{F}(M_1) \to \mathcal{F}(M_2) \to \mathcal{F}(M_3)$ is also exact at $\mathcal{F}(M_1)$ and $\mathcal{F}(M_2)$.

(6) Let $A \in \mathcal{A}$. Show that $\mathrm{Hom}_{\mathrm{Ab}}(-, A)\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ and $\mathrm{Hom}_{\mathrm{Ab}}(A, -)\colon \mathcal{A} \to \mathrm{Ab}$ are left-exact.

(7) Let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be an exact functor. Show that $\mathcal{F}$ is conservative if and only if $0 \in \mathcal{A}$ is the only object such that $\mathcal{F}(0) \simeq 0$.

(8) Let $\mathcal{F}\colon \mathrm{LMod}_A \to \mathrm{LMod}_B$ be a left/right adjoint. Show that $\mathcal{F}$ is right/left exact.

(9) Let $\mathcal{G}\colon \mathcal{A} \to \mathcal{B}$ be an additive/left-exact/right-exact/exact functor and $\mathcal{G}\colon \mathcal{A} \to \mathcal{B}$ a retract in $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$. Show that $\mathcal{G}$ is also additive/left-exact/right-exact/exact.

5.4 Exercise (Exercises on Vector Spaces)**.** Let $\mathcal{C}$ be the category containing the natural numbers $\mathbb{N}_0$ as objects and with Matrices as Hom-Set: $\mathrm{Hom}_{\mathcal{C}}(n, m) := K^{m \times n}$. Then composition of morphisms is matrix multiplication.

(1) Observe that $\mathcal{C}$ is a well-defined category.

(2) Show that $\mathcal{C}$ is equivalent to the full subcategory of finite dimensional k-vector spaces $\mathrm{FinVect}_k$[22].

(3) Show that the functor $\mathrm{Hom}_{\mathrm{LMod}_k}(-, k)\colon \mathrm{FinVect}_k^{\mathrm{op}} \to \mathrm{FinVect}_k$ is an equivalence of categories.

(4) Let $\mathcal{F} : \mathrm{LMod}_k \to \mathcal{A}$ be an additive functor. Prove or disprove that $\mathcal{F}$ is exact/limit preserving/colimit preserving.

(5) Let V be a k-vector space: Show that
   (a) $\mathrm{Hom}_{\mathrm{LMod}_k}(-, V)\colon \mathrm{LMod}_k \to \mathrm{Ab}$ is exact,
   (b) $\mathrm{Hom}_{\mathrm{LMod}_k}(V, -)\colon \mathrm{LMod}_k \to \mathrm{Ab}$ is exact,
   (c) $\mathrm{Hom}_{\mathrm{LMod}_k}(V, -), \mathrm{Hom}_{\mathrm{LMod}_k}(V, -)$ are conservative if and only if $V \neq 0$.

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[22] "Fundamental Theorem of Linear Algebra 1" - Fabien Morel

(6) Show that the free A-module functor $\mathrm{Free}\colon \mathrm{Set} \to \mathrm{LMod}_A$ is essentially surjective if and only if A is a skew-field.

(7) Let $\mathcal{F}\colon \mathrm{Ab} \to \mathrm{Ab}$ be an additive functor. Show that $\mathcal{F}$ restricts as
   (a) $\mathcal{F}\colon \mathrm{LMod}_{\mathbb{F}_p} \to \mathrm{LMod}_{\mathbb{F}_p}$
   (b) $\mathcal{F}\colon \mathrm{LMod}_{\mathbb{Q}} \to \mathrm{LMod}_{\mathbb{Q}}$

(8) Show that any additive functor $\mathcal{F}\colon \mathrm{LMod}_k \to \mathcal{A}$ or $\mathcal{F}\colon \mathrm{LMod}_k^{\mathrm{op}} \to \mathcal{A}$ is an exact functor.

(9) Let $\mathrm{Mono}(\mathcal{A}), \mathrm{Epi}(\mathcal{A}) \subseteq \mathrm{Ar}(\mathcal{A})$ be the full subcategories containing monomorphisms and epimorphisms resp. as objects. Show that there exists an equivalence of categories $\mathrm{Mono}(\mathcal{A}) \simeq \mathrm{Epi}(\mathcal{A})$.

6 Some Lawvere Theory Calculations

Let $\mathcal{C}$ be a category. Then the yoneda embedding $\mathcal{C} \xrightarrow{よ} \mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ identifies the category $\mathcal{C}$ with a subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$, the essential image of the yoneda embedding. It follows, that the essential image of the yoneda embedding contains only functors $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, which preserve limits. Now let $\mathcal{C}_0^{\mathrm{op}} \xhookrightarrow{\iota} \mathcal{C}^{\mathrm{op}}$ be a full subcategory, which is closed under finite products. Then we may precompose to get a functor $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \xrightarrow{\iota^*} \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{Set})$. Write $\mathrm{Fun}^{\prod}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{Set})$ for the full subcategory containing finite product preserving functors. Any object in the image of $\mathcal{C}$ under the composition $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{Set})$ will then preserve finite products, i.e., land in $\mathrm{Fun}^{\prod}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{Set})$. Our goal will be, to identify $\mathcal{C} \in \{\mathrm{Grp}, \mathrm{Ab}, \mathrm{Ring}, \mathrm{CRing}, \mathrm{LMod}_A, ...\}$ with $\mathrm{Fun}^{\prod}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{Set})$ for a suitable full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$.

6.1 Definition (Idempotent Complete Category)**.**

(1) Let f be a morphism in a category $\mathcal{C}$. Then f is called idempotent complete, if $f \circ f = f$.

(2) An idempotent morphism $f : c \to c$ is said to be split-idempotent, if there exists a commutative diagram

$$\begin{tikzcd}
c \arrow[r, "h"] \arrow[rr, bend left, "f"] & d \arrow[r, "g"] \arrow[rr, bend right, "\mathrm{id}_d"'] & c \arrow[r, "h"] & d
\end{tikzcd}$$

(3) A category $\mathcal{C}$ is said to be idempotent-complete, if every idempotent splits.

6.2 Definition (Idempotent Completion)**.** Let $\mathcal{C}$ be a category. Then an idempotent-completion of $\mathcal{C}$ is an idempotent complete category $\mathcal{C}^{\natural}$ together with a functor ι :

$\mathcal{C} \to \mathcal{C}^\natural$ such that $\iota^*\colon \mathrm{Fun}(\mathcal{C}^\natural, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is an equivalence of categories for an idempotent-complete category $\mathcal{D}$.

6.3 Remark. The idempotent completion is essentially a left-adjoint object for the inclusion of idempotent-complete categories into all categories.

6.4 Exercise.

(1) Show that the idempotent-completion is unique up to equivalence of categories.

(2) Let $\mathcal{C}$ be a category. Show that the idempotent completion is – up to equivalence – given by the smallest full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ containing the essential image of the yoneda embedding and which is closed under retracts.

6.5 Exercise. Let $\mathcal{C}$ be a 1-category[23] with finite limits or colimits. Show that $\mathcal{C}$ is idempotent complete.

6.6 Definition (Split Coequalizer)**.** Let $\Delta_{\leq 1}$ be the category containing the linear ordered sets $[0] = \{0\}, [1] = \{0, 1\}$ with ≤ 1 and order-preserving maps as morphisms. Then a Split coequalizer is a colimit over $\Delta^{\mathrm{op}}_{\leq 1}$ or less formal, over the category

$$\begin{tikzcd}
1 \arrow[r, shift left] \arrow[r, shift right] & 0 \arrow[l, bend right]
\end{tikzcd}$$

6.7 Definition (1-Strongly Finitely Presented Object)**.** Let $\mathcal{C}$ be a category and c an object in $\mathcal{C}$. Then c is called 1-strongly finitely presented (or 1-sfp) if $\mathrm{Hom}_{\mathcal{C}}(c, -)$ preserves filtered colimits and split coequalizers. We write $\mathcal{C}^{1-\mathrm{sfp}}$ for the full subcategory of 1-sfp objects.

6.8 Remark. Note that a 1-sfp object is by definition also a compact object.

6.9 Exercise.

(1) Do exercise 3.7 if you have not done them already.

(2) Let c be a 1-sfp object such that $c \simeq \mathrm{colim}_{\Delta^{\mathrm{op}}_{\leq 1}} c_i$. Show that c is a retract of $c_{[0]}$ or $c_{[1]}$.

6.10 Theorem (1-Strongly Finitely Presented Objects are closed under finite Coproducts)**.** *Let c_1, c_2 be 1-sfp objects in $\mathcal{C}$. Then so is $c_1 \coprod c_2$, if the coproduct exists.*

6.11 Exercise (1-Strongly Finitely Presented Objects)**.**

(1) Show that the 1-strongly finitely presented objects in Set are precisely the finite sets.

(2) Show that the 1-strongly finitely presented objects in Mon are precisely the retracts of finite free Monoids.

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[23] notably, a finitely bicomplete infinity category is in general not idempotent complete.

(3) Show that the 1-strongly finitely presented objects in CMon are precisely the retracts of finite free commutative Monoids.
(4) Show that the 1-strongly finitely presented objects in Grp are precisely the retracts of finite free groups.
(5) Show that the 1-strongly finitely presented objects in Ab are precisely the retracts of finite free abelian groups.
(6) Show that the 1-strongly finitely presented objects in LMod_A are precisely the retracts of finite free A-modules
(7) Show that the 1-strongly finitely presented objects in Ring are precisely the retracts of finite free non-commutative polynomial rings.
(8) Show that the 1-strongly finitely presented objects in CRing are precisely the retracts of finite free commutative polynomial rings.
(9) Show that the 1-strongly finitely presented objects in Alg_A are precisely the retracts of finite free A-Algebras.
(10) Show that the 1-strongly finitely presented objects in CAlg_A are precisely the retracts of finite free commutative A-Algebras.

6.12 Exercise. Show that each each category $\mathcal{C}$ in exercise 6.11 is equivalent to $\mathrm{Fun}^{\prod}((\mathcal{C}^{1-\mathrm{sfp}})^{\mathrm{op}}, \mathrm{Set})$ where $\mathcal{C}^{1-\mathrm{sfp}}$ is the full subcategory containing 1-strongly finitely presented objects and $\mathrm{Fun}^{\prod}(-,-)$ denotes the full subcategory consisting of product preserving functors.

We give a toy example, how knowing about this[24] can simplify your life. Let M be a set, then the power set $\mathcal{P}(M)$ admits a ring structure: the addition is the symmetric difference $N_1 \Delta N_2 := (N_1 \backslash N_2) \coprod (N_2 \backslash N_1)$ and multiplication is the intersection. Then M is the multiplicative unit and $\emptyset$ the additive unit.

6.13 Exercise. Use the equivalence $\mathrm{CAlg}_{\mathbb{F}_2} \simeq \mathrm{Fun}^{\prod}((\mathrm{CAlg}_{\mathbb{F}_2}^{\mathrm{fin,free}})^{\mathrm{op}}, \mathrm{Set})$ and the fact that $\mathrm{Hom}_{\mathrm{Set}}(M, -) : \mathrm{Set} \to \mathrm{Set}$ preserves products to show that $(\mathcal{P}(M), \Delta, \cap)$ is a commutative $\mathbb{F}_2$-Algebra.

7 Tensor Product of Modules

7.1 Definition (Tensor Product for Abelian Groups)**.** Let M be an abelian group. Then the tensor product

$$M \otimes_{\mathbb{Z}} - : \mathrm{Ab} \to \mathrm{Ab} \tag{1}$$

is defined as left adjoint to $\mathrm{Hom}_{\mathrm{Ab}}(M, -) : \mathrm{Ab} \to \mathrm{Ab}$.

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[24] These exercises are written with (nonabelian) derived functors, or Animation in mind

7.2 Remark. Observe that there exists a functor

$$\begin{align} \mathrm{Hom}_{\mathrm{Ab}}(-,-) : \mathrm{Ab} \to& \mathrm{Fun}(\mathrm{Ab},\mathrm{Ab})^{\mathrm{op}} \tag{2}\\ M \mapsto& \mathrm{Hom}_{\mathrm{Ab}}(M,-)) \tag{3} \end{align}$$

In particular, any group homomorphism $\varphi : M \to N$ induces a natural transformation $\mathrm{Hom}_{\mathrm{Ab}}(N,-) \xrightarrow{\varphi^*} \mathrm{Hom}_{\mathrm{Ab}}(M,-)$. Then passing to adjoints is somewhat functorial[25], i.e., let $\mathcal{L}_i \dashv \mathcal{R}$ be two adjunctions and $\psi^L : \mathcal{L}_1 \Rightarrow \mathcal{L}_2$ a natural transformation. Then passing to right adjoints gives an essentially unique natural transformation $\psi^R : \mathcal{R}_2 \Rightarrow \mathcal{R}_1$. This allows us to upgrade the tensor product to a functor

$$\begin{align} - \otimes_{\mathbb{Z}} - : \mathrm{Ab} \times \mathrm{Ab} \to& \mathrm{Ab} \tag{4}\\ (M,N) \mapsto& M \otimes_{\mathbb{Z}} N \tag{5} \end{align}$$

7.3 Exercise.

(1) Let $\varphi : A \to B$ be a ring homomorphism and P a projective module. Show that $B \otimes_A P$ is a projective module.
(2) Show that there exists an abelian group A such that the tensor product $- \otimes_{\mathbb{Z}} A$ is not left exact.
(3) Find a nontrivial abelian group A such that $A \otimes_{\mathbb{Z}} A \cong 0$.
(4) Find a nontrivial abelian group A such that
    (a) $A \otimes_{\mathbb{Z}} \mathbb{Q} \cong 0$
    (b) $A \otimes_{\mathbb{Z}} \mathbb{F}_p \cong 0$ for all primes p.

7.4 Exercise (Relative Tensor Product is (Split) Coequalizer)**.** Let A be a commutative ring and M, N a A-modules. Let $\eta_M : M \otimes_{\mathbb{Z}} A \to M, \eta_N : A \otimes_{\mathbb{Z}} N \to N$ be the multiplication maps. Show that the tensor product $M \otimes_A N$ is the colimit[26] of

$$\begin{tikzcd} M \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} N \arrow[r, shift left, "\mathrm{id}\otimes_{\mathbb{Z}}\eta_N"] \arrow[r, shift right, "\eta_M\otimes_{\mathbb{Z}}\mathrm{id}"'] & M \otimes_{\mathbb{Z}} N \end{tikzcd}$$

8 Finitely generated abelian groups

8.1 Theorem (Classification of finitely generated Modules over a PID)**.** *Let A be a PID and M a finitely generated module over A. Then there exists an isomorphism $M \cong \bigoplus_{i=1}^n A/(r_i)$ (where a_i may be zero).*

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[25] To be precise: Let $\mathrm{Cat}^L, \mathrm{Cat}^R$ be the subcategory of the $2,1$-category of $1,1)$-categories containing left (right) adjoints as morphisms. Then there exists an equivalence of categories $\mathrm{Cat}^L \simeq (\mathrm{Cat}^R)^{\mathrm{op}}$.

[26] even split coequalizer

8.2 Exercise.

(1) Let M be a finitely generated group with $M \otimes_{\mathbb{Z}} \mathbb{F}_p \cong 0$ for all primes p. Show that $M \cong 0$.

(2) Let M be a finitely generated abelian group. Show that the rank of M is determined by the dimensions of $M \otimes_{\mathbb{Z}} \mathbb{F}_p$.

(3) Let $\varphi\colon M \to N$ be a group homomorphism between finitely generated groups. Suppose $\varphi \otimes_{\mathbb{Z}} \mathbb{F}_p$ is an isomorphism for all primes. Show that $\varphi \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism.

(4) prove or disprove: suppose M, N are abelian groups and $\varphi \otimes_{\mathbb{Z}} \mathbb{F}_p \colon M \otimes_{\mathbb{Z}} \mathbb{F}_p \to N \otimes_{\mathbb{Z}} \mathbb{F}_p$ is an isomorphism for all primes p. Then φ is already an isomorphism.

9 Divisible Groups and Injective Modules

Throughout this section, $\mathcal{A}$ and $\mathcal{B}$ will denote abelian categories. But you can also assume that $\mathcal{A}, \mathcal{B} \in \{\mathrm{LMod}_A, \mathrm{LMod}_A^{\mathrm{op}}\}$.

9.1 Definition. Let M be an A-module and $a \in A$ a non zero-divisor. We call M (uniquely) a-divisible if

$$\begin{aligned} a\cdot\colon M &\to M \\ m &\mapsto a \cdot m \end{aligned}$$

is a surjection (isomorphism)[27]. We call M (uniquely) divisible if it is (uniquely) a-divisible for all non zero-divisors a.

9.2 Exercise.

(1) for A in Ab, the following are equivalent:
  (a) A is a $\mathbb{Q}$-vector space
  (b) A is uniquely divisible
  (c) A is torsion-free and divisible

(2) Let M be a divisible A-module, $N \subseteq M$ a submodule. Show that M/N is also divisible.

(3) Suppose that M is uniquely divisible. Prove or disprove, that M/N is also uniquely divisible.

(4) Conclude that $\mathbb{Q}/\mathbb{Z}$ is divisible.

9.3 Exercise. Let A be an integral domain.

(1) Show that the following are equivalent:

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[27] Observe that this is in general a map of sets and only one of A-modules if a lies in the center of A.

    (a) A is divisible,
    (b) A is uniquely divisible,
    (c) A is a field.
(2) Let $M \neq 0$ be a divisible A-module. Show that any nontrivial A-linear map $\varphi : M \to A$ is surjective.
(3) Let $M \neq 0$ be a divisible A-module. Show that the following are equivalent:
    (a) A is a field,
    (b) $\mathrm{Hom}_{\mathrm{LMod}_A}(M, A) \neq 0$.

Let $M \neq 0$ be a divisible, projective A-module. Show that A is a field.

9.4 Definition. An object A in $\mathcal{A}$ is called injective if $\mathrm{Hom}_{\mathcal{A}}(-, A)\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ is an exact functor. Dually, an object A in $\mathcal{A}$ is called projective if $\mathrm{Hom}_{\mathcal{A}}(A, -)\colon \mathcal{A} \to \mathrm{Ab}$ is an exact functor.

9.5 Exercise (Characterizations of Injective Objects)**.**
(1) Show that an object A is injective in $\mathcal{A}$ if and only if A is projective in $\mathcal{A}^{\mathrm{op}}$.
(2) Show that an object A is injective if and only if for any monomorphism $f\colon B \hookrightarrow B'$ and any morphism $g\colon B \to A$, there exists an extension

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, hook, "f"] & A \\
B' \arrow[ur, dashed, "g'"] &
\end{tikzcd}$$

(3) Show that an object A is injective if and only if each monomorphism $A \hookrightarrow B$ is a split mono.
(4) Show that an object A is injective if and only if each short exact sequence $A \hookrightarrow B \twoheadrightarrow C$ splits.

9.6 Exercise.
(1) Let $\mathcal{A}$ be a category, such that every object A in $\mathcal{A}$ is projective/injective. Show that each object in $\mathcal{A}$ is also injective/projective.
(2) Show that any module M over k is injective in LMod_k.
(3) Show that $\mathbb{F}_p$ is injective in $\mathrm{LMod}_{\mathbb{F}_p}$ but not in Ab.
(4) Show that injective objects are closed under arbitrary products.
(5) Show that $\mathbb{Z}$ is not injective as abelian group.

9.7 Theorem (Baer's Theorem for abelian groups, Lemma 05NU)**.** *An A-module M is injective if and only if for each submodule (i.e., left ideal) $\mathfrak{a} \subseteq A$ and each* *morphism* $\mathfrak{a} \to M$*, there exists an extension*

$$\begin{tikzcd}
\mathfrak{a} \arrow[r] \arrow[d] & M \\
A \arrow[ur, dashed] &
\end{tikzcd}$$

9.8 Corollary. *Let* A *be a PID. Then an* A*-module* M *is injective if and only if it is divisible.*

9.9 Corollary. *An abelian group* A *is injective if and only if it is divisible.*

9.10 Exercise.

(1) prove corollary 9.8 and corollary 9.9 using theorem 9.7.
(2) Let A be an integral domain and M an injective A-module. Show that M is divisible.
(3) Let A be an integral domain and $\mathcal{F} : \mathrm{LMod}_A \to \mathrm{Ab}$ a right exact functor. Show that $\mathcal{F}$ preserves injective objects.
(4) Let A be an integral domain and $M \neq 0$ a projective and injective A-module. Show that A is a field.
(5) Show that injective abelian groups are closed under arbitrary colimits.
(6) prove or disprove that the functors
    (a) $\mathrm{Hom}_{\mathrm{Ab}}(-, \mathbb{Z})\colon \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}$
    (b) $\mathrm{Hom}_{\mathrm{Ab}}(-, \mathbb{Q})\colon \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}$
    (c) $\mathrm{Hom}_{\mathrm{Ab}}(-, \mathbb{Q}/\mathbb{Z})\colon \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}$

    are conservative.
(7) Let FinAb be the full subcategory of finite abelian groups. Show that the functor $\mathrm{Hom}_{\mathrm{Ab}}(-, \mathbb{Q}/\mathbb{Z})\colon \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}$ restricts to an equivalence of categories $\mathrm{Hom}_{\mathrm{Ab}}(-, \mathbb{Q}/\mathbb{Z})\colon \mathrm{FinAb}^{\mathrm{op}} \to \mathrm{FinAb}$.
(8) Let FinFreeAb be the full subcategory of finitely generated, free abelian groups. Show that the functor $\mathrm{Hom}_{\mathrm{Ab}}(-, \mathbb{Z})\colon \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}$ restricts to an equivalence of categories

$$\mathrm{Hom}_{\mathrm{Ab}}(-, \mathbb{Z})\colon \mathrm{FinFreeAb}^{\mathrm{op}} \to \mathrm{FinFreeAb}$$

(9) Let FinGenAb be the full subcategory of finitely generated abelian groups. Show that there does not exist an equivalence $\mathrm{FinGenAb} \simeq \mathrm{FinGenAb}^{\mathrm{op}}$.
(10) Let A be an abelian group. Show that there exists an injection $A \hookrightarrow B$ for an injective group B.
(11) Show that any injective abelian group A is a retract of $\prod_I \mathbb{Q}/\mathbb{Z}$ for some I.
(12) Let M be an injective A-module and $\varphi\colon A \to B$ a ring homomorphism. Show that $\mathrm{Hom}_{\mathrm{LMod}_A}(B, M)$ is an injective B-module.

(13) Let M be an A-module. Show that there exists an injection $M \hookrightarrow N$ for an injective A-module N.

(14) Let M be an A-module. Show that there exists a sequence $0 \hookrightarrow M \hookrightarrow N_1 \to N_2 \to N_3 \to \ldots$ which is exact at M and N_i and where N_i is an injective module.

(15) (*) Let A be an abelian group such that the canonical projection $\mathrm{pr}\colon \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ induces an isomorphism $\mathrm{pr}_*\colon \mathrm{Hom}_{\mathrm{Ab}}(A, \mathbb{Q}) \to \mathrm{Hom}_{\mathrm{Ab}}(A, \mathbb{Q}/\mathbb{Z})$. Show that A is the zero group[28].

9.11 Exercise (Ext of finitely generated abelian groups)**.**

(1) Let $\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n$ be the evident exact sequence. Then compute the cokernel of
   (a) $\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{(\cdot n)^*} \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, \mathbb{Z})$
   (b) $\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{(\cdot n)^*} \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$

   where $(\cdot n)^*$ is precomposition with the map $\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$.

(2) Let A be a finitely generated abelian group. Choose an exact sequence $\mathbb{Z}^k \xrightarrow{f} \mathbb{Z}^l \to A$ [29] and compute the cokernel of
   (a) $\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}^l, \mathbb{Z}) \xrightarrow{f_*} \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}^k, \mathbb{Z})$
   (b) $\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}^l, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{f_*} \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}^k, \mathbb{Z}/m\mathbb{Z})$

   where f_* is postcomposition with the map $\mathbb{Z}^k \xrightarrow{\cdot n} \mathbb{Z}^l$

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[28] this essentially shows that the derived hom functor $R\mathrm{Hom}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})\colon \mathcal{D}(\mathbb{Z})^{\mathrm{op}} \to \mathcal{D}(\mathbb{Z})$ is conservative

[29] this is a projective resolution of length 2